Study of an adaptive anti-windup strategy for cross-directional control systems

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Abstract

The cross directional control problem of web forming processes subject to input constraints is investigated. We focus our discussion on the analysis of the robust properties of a recently proposed anti-windup strategy for cross-directional control systems, which is based on a spatial frequency decomposition of the system’s interaction matrix. The strategy effectively deals with the problem of having only limited control authority by prioritizing the control directions in which the control effort is employed. This property of the strategy prevents the controller from utilizing the available control energy to compensate the disturbance in control directions which are difficult to control i.e., in those directions associated with a small gain. Simulation results show that the attained steady state performance of the strategy is superior to that achieved by traditional MIMO anti-windup schemes and is close to an optimal QP solution to the problem.

1 Introduction

Constrained control is a well established area in control theory. Its development has reported important benefits when applied to processes subject to hard input or state constraints (De Doná et al., 2002). All real system are ultimately limited by physical constraints which impose a lower bound on the performance of control strategies applied to the process. Moreover, if constraints are ignored in the control design stage, then unnecessary performance degradation of the control loop may occur.

In this paper, our focus our is on high dimensional systems subject to input constraints. This type of system is especially challenging to control in the presence of constraints, since the strong interaction between variables that usually characterize them, makes them prone to poor conditioning. As a result, high dimensional systems can be characterized by having several control directions that are hard to control, in the sense that a considerable control effort is necessary in order to compensate for a disturbance acting in one or more of those directions. In the presence of constraints, this inherent characteristic of poorly
conditioned systems is likely to cause the manipulated variables to be operating on the constraint boundary during significant time periods. Therefore, important performance degradation may be expected.

Web forming processes represent a well known example of high dimensional systems. They can be found in many practical applications, especially in the paper-making, plastic and metal industries (Featherstone et al., 2000; Goodwin et al., 1990). Web forming processes usually have two main control objectives: to regulate the sheet property profile along the machine, referred to as the **machine-direction** (MD), and to maintain a flat profile across the machine web, referred to as the **cross-direction** (CD).

We have recently proposed an adaptive control strategy for high dimensional cross-directional control systems aimed at overcoming some of the difficulties described above (Goodwin et al., 2002). The strategy relies on a Singular Value Decomposition of the system’s interaction matrix and it effectively deals with the problem of having only limited control authority, by prioritizing the control directions in which to perform control. This approach is an extension of existing ideas for anti-windup control and we have named it the Spatial Frequency Anti-Windup (SFAW) strategy.

In this paper we analyse the adaptive properties of the SFAW strategy and we show that the performance is in general superior than the performance obtained by other proposed schemes, even when the model is known.

## 2 The cross-directional control problem

The cross-directional control problem for web forming processes exhibits several difficulties:

- the system is usually of very high dimension, with the number of actuators ranging between 30 and possibly up to 300, and sensor measurements between 200 and 1000 (Stewart et al., 1999),

- there exists strong coupling between the steady state effect of the actuators over the system outputs which results in poor conditioning,

- the actuators usually have limited authority. Hence actuator saturation is very likely to occur (from this point of view, web forming processes can be thought of being as similar to under-actuated systems, since there exist spatial components of the output space that are not controllable in any significant practical sense due to input constraints).

In many applications of cross directional control, it is generally assumed that the process can be modeled by separating its spatial response from its dynamical response (Featherstone et al., 2000):

\[
y(s) = g(s) Mu(s),
\]

where \( y(s) \in \mathbb{R}^N \) is the vector of sensor measurements and \( u(s) \in \mathbb{R}^N \) is the vector of control signals. \( M \in \mathbb{R}^{N \times N} \) is a constant and real valued matrix, known as the normalized d.c. **interaction matrix.** \( M \), as its name suggests, represents the spatial influence of each actuator on the system outputs. Even though in many practical applications \( M \) can be
non square, we restrict the analysis here to cases in which we have equal numbers of
input and output signals (natural extensions exist to the non-square case). In Equation
(1), \( g(s) \) is a scalar LTI transfer function which accounts for the dynamic behaviour
of the process. This, in turn, is assumed to be the same along the cross direction of
the machine. Typically, \( g(s) \) can be taken to be a first order, stable and minimum phase
transfer function, having a transport delay \( T_d > 0 \) (Stewart, 2000; Bergh and MacGregor,
1987):

\[
g(s) = \frac{e^{-T_d s}}{\tau s + 1} \tag{2}
\]

where \( \tau > 0 \).

The control objective of cross-directional control systems is usually stated in terms
of the steady state variance of the cross-directional profile of the web:

\[
\min_{u \in U} \| y \|_2^2 \tag{3}
\]

where the vector of manipulated variables \( u(t) \) is forced to belong to the constraint polytope \( U \):

\[
U = \{ u(t) \in \mathbb{R}^N \mid \| u(t) \|_\infty \leq \Delta \} \tag{4}
\]

Due to the symmetries inherent in the physical architecture of web forming processes, the interaction matrix \( M \) usually has the structure of a Toeplitz symmetric matrix
(Featherstone et al., 2000; Featherstone and Braatz, 1997). In other specific cases, the
structure of \( M \) is that of a circulant symmetric matrix i.e., each row of \( M \) is a shifted
version of the previous row, where its first entry is equal to the last entry of the previous row.
Clearly, such a description for \( M \) is appropriate for processes that have a circulant structure. This is the case, for example, in paper machines where edge effects are neglected,
dies for plastic films and multi-zone crystal growth furnaces (Hovd et al., 1997).

The simple model structure (1) suggests that the system could be easily decoupled by
introducing the inverse of \( M \) in the controller definition. However, direct application
of this approach usually leads to numerical problems since, as pointed out in Section 1, large
scale systems are generally poorly conditioned. (Duncan and Bryant, 1997). A natural
way of dealing with this difficulty is to consider a Singular Value Decomposition (SVD)
of the interaction matrix \( M \) (Wellstead et al., 2000; Stewart, 2000). This is indeed the
standard approach for cross-directional control problems (Featherstone et al., 2000) and
it includes the more general case in which \( M \) is not square. Thus, \( M \) can be decomposed
as follows:

\[
M = U S V^T \tag{5}
\]

where \( U, S \) and \( V \) are all real valued matrices of dimension \( N \times N \). \( S \) is a diagonal matrix
containing the singular values of \( M \) arranged in decreasing order. Moreover, \( U \) and \( V \) are
orthogonal matrices, whose columns generate an orthonormal base for the system output
and input space respectively, that is \( UU^T = VV^T = I \) and \( U^T U = I \) and \( VV^T = V^T V = I \).

Featherstone and Braatz (1998) have suggested a slight modification of the decompo-
sition in (5) where the entries of \( S \) are allowed to have a sign and they are referred to as
pseudo singular values (Featherstone and Braatz, 1998; Featherstone and Braatz, 1997;
Hovd et al., 1997; Featherstone and Braatz, 1995). This alternative description allows the
extension to the MIMO case of some of the results well known in the SISO case. For ex-
ample in the scalar case it is clear that the sign of the plant d.c. gain needs to be correctly
identified for the corresponding closed loop to be stable; otherwise this would be equivalent to having positive feedback in the control loop instead of negative feedback. For large scale systems a similar conclusion can be drawn in terms of the associated pseudo singular values. Indeed, if the sign of any pseudo-singular value of the system is not known with confidence, then no internal model controller can stabilize the system even if complete knowledge of $U$ and $V$ is assumed (Featherstone and Braatz, 1998).

Closely related to the SVD of $M$ proposed in (5) is the eigenvalue-eigenvector decomposition of a symmetric matrix (indeed, if $M$ is symmetric and positive definite, both decompositions are equivalent (Strang, 1988)). This approach is especially useful when $M$ is circulant, since any circulant matrix $C \in \mathbb{C}^{N \times N}$ has a complete set of independent eigenvectors. Moreover, every circulant matrix $C$ of the same order $N$ can be diagonalized by the same eigenvector matrix, namely the Fourier matrix (Davis, 1979):

$$W^H = \frac{1}{\sqrt{N}} \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & w & w^2 & \cdots & w^{N-1} \\
1 & w^2 & w^4 & \cdots & w^{N-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & w^{N-1} & w^{N-2} & \cdots & w
\end{bmatrix}; \quad (6)$$

where $w = e^{j \frac{2\pi}{N}}$. Hence:

$$C = W^H \Lambda_C W \quad (7)$$

and $\Lambda_C$ is a diagonal matrix containing the eigenvalues of $C$. When a circular matrix $C$ is symmetric, the analysis can be simplified further, since the associated eigenvector matrix can be chosen to be the so called Real Fourier matrix $R$ (Davis, 1979; Hovd and Skogestad, 1994; Stewart, 2000). Hence:

$$C = R^T \Lambda_C R \quad (8)$$

This is a result of the fact that the eigenvalues of $C$ appear in pairs, hence the corresponding eigenvectors can be chosen to be real.

The diagonalization of $C$ through $W$ and $R$ in (7) and (8) suggests a direct connection between the singular values (or the absolute value of the eigenvalues) of a circulant (symmetric) interaction matrix $M$ and the Discrete Fourier Transform of the weighting sequence $\{m_0, m_1, \ldots m_{N-1}\}$ representing the steady state effect of one single actuator over the entire set of outputs. Indeed, it can be shown that (Goodwin et al., 2002):

$$\sigma_l = |\lambda_l| = |\hat{m}_l|, \quad \text{for } l = 0, \ldots, N - 1. \quad (9)$$

where $\hat{m}_l$ is the DFT of the weighting sequence $i.e.$,

$$\hat{m}_l = \sum_{k=0}^{N-1} m_k w^{-lk} \quad (10)$$

with $w = e^{j \frac{2\pi}{N}}$.

Equation (9) shows that the singular values of a circulant (symmetric) matrix $M$ can be directly computed from the DFT of the weighting sequence $\{m_0, m_1, \ldots m_{N-1}\}$. This, in turn, allows the use of a discrete frequency analysis approach to estimate the qualitative behaviour of the singular values of $M$. 

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Remark 2.1. Note that both matrices $W$ and $R$ are independent of the actual entries in the matrix $M$; indeed, both can be easily computed based on the knowledge of the dimension $N$ of $M$. This is certainly a very useful feature of circulant (symmetric) matrices which allows a completely decoupled controller design and synthesis, even when model uncertainties are present. However, large scale circulant systems are still subject to robustness issues due to poor conditioning. In fact, as we have mentioned before, a non reliable estimation of the sign associated with the smallest pseudo singular values of $M$ can lead to the instability of the corresponding control loop.

3 The Spatial Frequency Anti-Windup (SFAW) strategy

Large scale systems that allow a Singular Value Decomposition as in (5) can be effectively dealt with by using a decoupling MIMO controller having the following structure:

$$C(s) = V C^*(s) U^T = V \text{diag}\{c_1^*(s), c_2^*(s), \ldots, c_N^*(s)\} U^T \quad (11)$$

Each scalar controller $c_i^*(s)$, $i = 1, \ldots, N$ can be designed separately, by means of any appropriate control design strategy. For example, in the case of the simple dynamics (2) associated with web forming processes, one may adopt a PI control law of the form:

$$c_i^*(s) = \frac{\hat{\tau} s + 1}{k_i s} \quad (12)$$

If special handling of the time delay $T_d$ is needed, then a Smith predictor can be an alternative choice.

A controller such as the one described in (11) and (12) will perform adequately, provided the system (1) is well behaved and the manipulated variable $u(t)$ is not subject to any constraint. However, as described in Section 2, this is usually not the case for high dimensional systems such as those analysed here.

The robustness issue raised by the poor conditioning of high dimensional cross-directional systems has been extensively studied in earlier contributions; see, for example, (VanAntwerp et al., 2001; Featherstone and Braatz, 1998; Duncan, 1995; Laughlin et al., 1993). In terms of the uncertainty associated with the value and sign of the smallest pseudo singular values of the interaction matrix $M$, the generally accepted advise is to avoid controlling the plant in directions associated with uncertain gains. These, in turn, are referred to as “uncontrollable modes” (Heath, 1996; Duncan and Bryant, 1997). Therefore, if only a limited subset $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ with $r \leq N$ of the pseudo singular values of $M$ are known with reasonable confidence, then a robust controller structure would be similar to (11) with $C^*(s)$ defined as follows (Featherstone and Braatz, 1998):

$$C^*(s) = C_r^*(s) = \text{diag}\{c_1^*(s), \ldots, c_r^*(s), 0, \ldots, 0\} \quad (13)$$

where we see that the control action is effectively disabled in the “uncontrollable” directions by setting the corresponding controller gains to zero; of course this does not explicitly account for controller saturation.

The most common approach for dealing with high dimensional cross-directional control systems subject to constraints is undoubtedly Model Predictive Control (see, for example (VanAntwerp et al., 2001) and references therein). However, the main drawback
is that MPC can be computationally demanding, especially when dealing with a system with possibly up to 200 actuators and sensor measurements. Also, it does not explicitly account for the other problem of gain uncertainty.

An alternative approach to handling input constraints is the design of a linear controller incorporating anti-windup protection. This is indeed a common practice in industry; its main advantage being that it is a simple technique to implement. Nevertheless, a MIMO anti-windup controller cannot be exempted from the inherent difficulties associated to the control of poorly conditioned systems. As we mentioned in Section 1, a disturbance acting in a control direction hard to control will generate a considerable excursion of the manipulated variable in that direction, leading invariably to input saturation. This effect commonly manifests itself in the observed phenomenon of picketing, with each actuator being driven to alternate saturation levels. As a result, important performance degradation may be expected. Simulations results to be presented in Section 5 will substantiate this assertion.

With this as background, we have recently proposed an adaptive control strategy (Goodwin et al., 2002) for high dimensional cross-directional control systems aimed at overcoming some of the difficulties described above. The strategy relies on a SVD decomposition of the system, such as the one in Equation (5), and it is an extension of existing ideas for anti-windup control. The main feature of this approach, which we have called the Spatial Frequency Anti-Windup (SFAW) strategy (Goodwin et al., 2002) is that, rather than fixing the number of active directions a priori (as would be the case when using a controller structure such as (13)), we allow the selection to be made on-line, by scaling back the manipulated variables in the low authority directions until actuator saturation is avoided. We envisage two key advantages of using the SFAW strategy:

- the choice of active directions is matched to the particular characteristic of the current disturbance,
- additional robustness features accrue due to the fact that there is an inherent feedback mechanism involved in making the choice.

The SFAW strategy fits into a general canonical structure which is common to many anti-windup schemes. These schemes can be constructed by rearranging the control law so that all of the control law dynamics are driven by the real (i.e. constrained) actuator signals (Goodwin et al., 2001). To describe the idea in more detail, assume, without loss of generality, that $C^*(s)$ in (11) is biproper (if this is not the case, additional 'fast' zeros can be appended). Thus, we can write:

$$C^*(s)^{-1} = C^*_\infty^{-1} + \tilde{C}^*(s)$$

(14)

where $C^*_\infty$ is the high frequency gain matrix i.e., $C^*_\infty = \lim_{s \to \infty} C^*(s)$, and $\tilde{C}^*(s)$ is a strictly proper transfer function. We also assume that the controller has a stable inverse, so that $C^*(s)$ in (14) is stable. We can then implement $C^*(s)$ using the feedback scheme shown in Figure 1. The scheme is clearly equivalent to $C(s)$ if the limiting circuit in the forward path is chosen to be the identity matrix. This is indeed the case when $u(t) = Vu^*(t)$ is inside the constraint polytope. On the contrary, if the demanded control vector $\hat{u}^*(t)$ causes $u(t)$ to come outside the constraint polytope, then the limiting circuit should operate in such a way that $Vu^*(t)$ is always equal to the actual control vector applied to
the system. This is the key feature that makes the configuration of Figure 1 incorporate anti-windup protection (Goodwin et al., 2001).

In the MIMO case the design of the limiting circuit in Figure 1 presents some subtleties since there may be different ways of bringing \( u(t) \) back inside the constraint polytope. Well known options are simple clipping the components of \( u(t) \) or scaling the control vector to preserve directionality. However, neither of those simple strategies is suitable for the current problem since they do not account for the issue of widely different gains.

The key characteristic of the proposed SFAW strategy is the specific way in which the limiting circuit determines \( u(t) \) so that \( u(t) \in U \). The way in which this is done is based on the fact that the singular value decomposition of \( M \) in (5) generates a natural order among the plant input directions in terms of the control effort necessary to control them. Indeed, in (5) the control directions are ordered starting from the easiest to control (associated with the biggest \( \sigma_i \)) and ending with the hardest to control (associated with the smallest \( \sigma_i \)).

Heuristically, it seems reasonable to use the limited control authority available in \( u(t) \) to first compensate for the disturbance components in the control directions that require less control energy. As such, we make sure that the available control authority is not "wasted" doing very little in the directions that are harder to control.

Stated more formally, the limiting circuit associated with the SFAW strategy defines a diagonal weighting matrix \( P_\gamma(t) \in \mathbb{R}^{N \times N} \) such that:

\[
\hat{u}(t) = P_\gamma(t) \hat{u}(t)
\]

where,

\[
P_\gamma(t) = \begin{cases} 
\text{diag}\{p_1, \ldots, p_r, p_{r+1}, 0, \ldots, 0\}, & \text{when } \|V\hat{u}^*(t)\|_\infty \geq \Delta \\
I_{N \times N}, & \text{when } \|V\hat{u}^*(t)\|_\infty < \Delta 
\end{cases}
\]

and \( \Delta \in \mathbb{R} \) is the saturation level imposed to every entry of \( u(t) \).

Moreover, \( \gamma(t) \in \mathbb{R} \) is a variable allowed to evolve only in the interval \([0, N]\) which symbolises the spatial components that are active at any given time \( t \). If \( r \in \mathbb{Z}^+ \) is the biggest integer such that \( r \leq \gamma \), then the entries of \( P_\gamma(t) \) in (16) are defined as follows:

\[
p_1 = p_2 = \cdots = p_r = 1
\]

\[
p_{r+1} = \gamma - r
\]

\[
\text{s.t. } \|u(t)\|_\infty = \|VP_\gamma(t) \hat{u}^*(t)\|_\infty = \Delta
\]
In summary, the SFAW strategy computes the value of $\gamma(t)$ in such a way that the control vector $u(t)$ is always on the border of the constraint polytope $U_c$. From (17)-(18) it is clear that $|p_i| \leq 1$ for $i = 1, 2, \ldots, r + 1$.

Some properties of the above scheme have been discussed in (Goodwin et al., 2002). A stability analysis is also presented, based on the application of the Circle Criterion.

**Remark 3.1.** Note that utilizing the SFAW strategy we overcome the problem of losing decoupling in the singular value space when the system is subject to input saturation. This is because the control vector $u(t)$ is always kept on the boundary of the constraint polytope no matter what control signal is the controller actually requesting.

## 4 Performance analysis of the SFAW strategy

In this section we analyse additional properties inherent in the proposed SFAW control. These are a direct result of the specific way in which the strategy achieves anti-windup protection. In particular, we will concentrate on the robustness features of the SFAW strategy, which represent, as we will see, a rather surprising added benefit of this approach.

As we have discussed in Section 2 the uncertainty in the sign of the smallest pseudo singular values of $M$ can lead to instability.

We analyse the robustness of the proposed scheme by first assuming that the rotation matrices $U$ and $V$ of (5) are completely known. This assumption is not as restrictive as it might seem initially. Indeed we have seen that every circulant symmetric matrix can be diagonalized by the same Real Fourier matrix $R$, so that no uncertainty is associated to $U$ and $V$. In addition, it is well known (Grenander and Szegö, 1958) that a matrix $M$ having a general Toeplitz structure can be approximated by a circulant matrix provided the non zero entries of $M$ are concentrated close to the main diagonal and $M$ is a matrix of high dimension.

For simplicity we restrict attention to the case in which the transfer function representing the dynamics of the cross-directional plant has no transport delay i.e., $T_d = 0$ and the time constant of the system is completely known i.e., $\hat{\tau} = \tau$ in (12). Thus we focus on the gain uncertainty.

It is obvious that the algorithm ensures that the inputs and outputs of the plant are bounded due to the stability of the plant and boundedness of the input. What is more interesting, though more difficult to analyse, is the performance achieved by the algorithm. The following theorem gives some preliminary results in this regard:

**Theorem 4.1.** Consider the application of the SFAW strategy to the web forming process (1) and let the closed loop system be perturbed by a given static disturbance vector $d$:

$$d = U d^* = \sum_{k=1}^{N} \hat{u}_k d^*_k$$

(19)

where $\hat{u}_k$ represents the $k$-th column of the matrix $U$ and $d^*_k$ is the $k$-th entry of the disturbance vector $d^*$ in the singular value space. Based on $d^*$, let us define the integer $r$ as follows:

$$r(d^*) = \max(i) \text{ s. t. } \|u_{d,i}\|_\infty < \Delta$$

(20)
where,

\[ u_{d,i} = \sum_{k=1}^{i} \tilde{v}_k \frac{-d_k^*}{\sigma_k} \]  \hfill (21)

Then, given any \( \epsilon > 0 \), there exists a finite time \( t_\epsilon > 0 \) such that the following bound on the square of the \( l_2 \) norm of the error in the output profile is achieved:

(i). If the \( r + 1 \) control loops in the singular value space have the correct sign of the corresponding pseudo singular value, then \( \forall t > t_\epsilon \):  

\[ \|e(t)\|_2^2 \leq \sum_{k=r+1}^{N} d_k^* + \epsilon \]  \hfill (22)

(ii). If the first \( r \) components have the right sign for the pseudo singular value but any of the loops from \( r + 1 \) onwards may have the wrong sign, then \( \forall t > t_\epsilon \):  

\[ \|e(t)\|_2^2 \leq \sum_{k=r+1}^{N} (|d_k^*| + |\tilde{\sigma}_k D_k^{\text{max}}|)^2 + \epsilon \]  \hfill (23)

where \( D_k^{\text{max}} \) is the maximum euclidean distance between \( \mathbf{u}_{d,r} \) and the boundary of the constraint polytope \( \mathcal{U} \) in the direction \( \tilde{v}_k \), i.e.,  

\[ D_k^{\text{max}} = \max(\beta_k) \text{ s. t. } \|\mathbf{u}_{d,r} + \beta_k \tilde{v}_k\|_{\infty} = \Delta \]  \hfill (24)

(iii). If among the first \( r \) loops there exist a set \( \{u_l^* \mid l \leq r\} \) of components which have the wrong sign for \( \tilde{\sigma}_{l_0} \) and \( l_0 \) represents the smallest index in that set, then \( \forall t > t_\epsilon \):  

\[ \|e(t)\|_2^2 \leq \sum_{k=l_0}^{N} (|d_k^*| + |\tilde{\sigma}_k D_k^{\text{max}}|)^2 + \epsilon \]  \hfill (25)

Proof. (i). We start by considering only the first loop in the sequence \( i = 1, \ldots, r \). We see that the scaling gain \( p_i \) defined by \( P_i(t) \) in (16) only depends on the dynamic evolution of \( u_i^*(t) \), because all other loop outputs have to be zero before we can adjust \( p_i \). This follows by construction from the definition of \( P_i(t) \) and implies that we are able to analyse the first loop in isolation. We note that the loop is characterized by a first order dynamics. It is therefore easily seen (Goodwin et al., 2002) that for any \( \epsilon_1 \), there exists a finite time \( t_1 \) such that \( \forall t > t_1 \) we have,

\[ |u_i^*(t) - \overline{u}_i^*| \leq \epsilon_1 \] \hfill (26)

\[ \overline{u}_i^* = \frac{-d_i^*}{\sigma_1} \] \hfill (27)

In other words, \( u_i^*(t) \) will be arbitrarily close to \( \overline{u}_i^* \) after a finite time. We then consider the second control loop \( i = 2 \). We claim that the situation is not worse than the situation in which this second loop is not able to run until time \( t = t_1 \). We can then apply an identical argument to the second loop as used for the first loop. Indeed, we can analyse the \( i \)-th loop in isolation provided we allow sufficient time.
for all previous loops to be near steady state. By induction, for any given \( \tau \), there exists a finite time \( t_\tau \) such that:

\[
|u_i^*(t) - \pi_i^*| \leq \tau \quad \text{for } i = 1, \ldots, r
\]  

(28)

Based on condition (20), we also know that by appropriate choice of \( \tau \), the sum of the squares of the first \( r \) components will be within \( \varepsilon \) of zero. The \((r + 1)\)-th loop may decrease the size of \( d_{r+1}^* \). However, in (22), we include all components from \( r + 1 \) on wards.

(ii). This follows as for part (i) save that now it is possible that any of the loops from \( r + 1 \) on wards may use the residual control authority (after the first \( r \) loops are essentially in steady state). Moreover, these loops act in a contrary direction to the disturbance. The expression (23) over-bounds the worst possible scenario.

(iii). This is as in (i) and (ii) save that we can only ensure that the first \( l_0 - 1 \) loops approach their steady state values. The bound is then similar to (ii) but beginning from the \( l_0 \)-th loop.

\[ \blacksquare \]

**Remark 4.1.** In essence Theorem 4.1 shows that the adaptive algorithm ultimately chooses (beginning from the largest singular value) all directions whose associated steady state controls, for the given disturbance, are feasible, unless it is stopped at some higher level in the singular value chain by the existence of an unstable loop.

We illustrate the above theorem by the following example:

**Example 4.2.** Consider the model (1) of the cross directional dynamic behaviour of a blown film process (Featherstone and Braatz, 1997) with \( \tau = 1[s] \). This type of process has an interaction matrix \( M \) with a circulant symmetric structure. Let \( N = 2n + 1 \), \( n \in \mathbb{Z}^+ \), and assume that the effect of one actuator over the system outputs is given by the following profile:

\[
m_{0,l} = \begin{cases} 
e^{-\alpha l} & \text{for } l = 0, 1, \ldots, n, \\
e^{-\alpha (N+1-l)} & \text{for } l = n + 1, \ldots, 2n. \end{cases}
\]

(29)

where \( \alpha > 0 \). In particular, take \( \alpha = 0.2 \) and \( n = 10 \). Since \( M \) is circulant and symmetric, the associated eigenvalues are real and equal to the pseudo singular values of \( M \). Moreover, they repeat themselves in pairs. Assume that the SFAW strategy has been designed based on a model of the process which has the incorrect sign for the last 4 pairs of pseudo singular values. The system subject to a certain step output disturbance \( d(t) \) at time \( t = 1[s] \). Figure 2 shows the time evolution of \( \gamma(t) \) computed by the SFAW strategy. We see that \( \gamma(t) \) reaches a steady state \( \overline{\gamma} \) even in the presence of the uncertainty in the sign of certain singular values. Also notice that \( \overline{\gamma} = 13.2 \), hence the 14-th loop, which is the first in the singular value chain to have the wrong sign, is the one scaled back with a factor 0.2.
5 Simulation results

In this section we compare the performance of the proposed SFAW strategy with other known MIMO anti-windup schemes. In particular, we consider the following 3 control strategies:

- Anti-windup controller with *input clipping*,
- Anti-windup controller with *input scaling*,
- SFAW strategy.

We assume that the three approaches listed above use the same controller (11) with the same tuning parameters for every scalar controller $c_i^*(s)$ in (12).

We also want to investigate and compare the performance of the strategies when a different type of uncertainty than the one analysed in Section 4 occurs, namely when there is a mismatch between the model structure used to synthesise the controller of Figure 1 and the actual structure of the system interaction matrix $M$. When this is the case we loose decoupling in the spatial space *i.e.*, the scalar controllers $c_i^*(s)$ interact with each other.

Say that the dimension $N$ of the matrix $M$ is equal to $2n + 1$ with $n \in \mathbb{Z}^+$. We then adopt the following possible structures for the system interaction matrix $M$:

- a circulant symmetric structure with exponential profile *i.e.*,
  \[ m_{i,j} = \begin{cases} 
  e^{-\alpha|i-j|} & \text{for } |i-j| \leq n, \\
  e^{-\alpha(N-|i-j|)} & \text{for } |i-j| > n
  \end{cases} \tag{30} \]

- a Toeplitz (non circulant) symmetric structure with exponential profile *i.e.*,
  \[ m_{i,j} = e^{-\alpha|i-j|} \tag{31} \]

Also, let the dynamical response of the system be characterized by a time constant $\tau = 1\,[s]$ in (2) and assume, for simplicity, that there is no time delay *i.e.*, $T_d = 0$. Also, let the interaction matrix $M$ of the system have dimension $N = 21$ and assume that the
manipulated variable vector $u(t)$ is forced to belong to the constraint polytope $U$ in (4) with $\Delta = 1$.

The cross-directional control system is perturbed with a given output disturbance having the same profile in each test. In Figure 3, we present the steady state response of the system using the different anti-windup MIMO controllers considered in the analysis. The thicker line corresponds to the applied disturbance profile. In (a) we see that the SFAW strategy significantly outperforms the other 2 anti-windup approaches. The strategy effectively uses the available control authority to cancel the easy to control "low spatial frequency" components. This performance result is also shown in (b) where the same steady state profiles have been plotted in the spatial frequency domain. The SFAW strategy achieves zero steady state error in the first 7 low spatial frequency components and its performance is comparable to the one obtained by input clipping anti-windup and input scaling anti-windup in the highest spatial frequency components.

We also show in Figure 4 the corresponding steady state profile of the actuator movements: both input clipping and input scaling anti-windup strategies are dominated by the high spatial frequency components which generates the well known picketing phenomenon reported in many real applications of web forming processes subject to hard constraints. This type of actuator profile is highly non desirable since it can cause severe

<table>
<thead>
<tr>
<th></th>
<th>Model known</th>
<th>Model mismatch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input clipping</td>
<td>3.32</td>
<td>3.79</td>
</tr>
<tr>
<td>Input scaling</td>
<td>2.21</td>
<td>2.19</td>
</tr>
<tr>
<td>SFAW</td>
<td>1.24</td>
<td>1.27</td>
</tr>
<tr>
<td>QP</td>
<td>0.98</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Comparison of the $l_2$ norm of the steady state profile achieved by different anti-windup control strategies.
damage to the actuator mechanisms. On the contrary, the actuator profile obtained with the SFAW strategy is smoother and only the 7-th actuator is forced on the constraint border.

Table 5 summarises the results obtained when a model mismatch between the actual plant and model occurs. The entries of the table represent the $l_2$ norm of the corresponding steady state profiles. The right column of the table shows the results when the interaction matrix $M$ has a Toeplitz symmetric but non circulant structure and the controller is designed using a circulant symmetric structure for $M$. We note that the variance of the output profiles do not differ markably between the case when complete knowledge of the plant is assumed and when model mismatch occurs. Moreover, in both conditions the variance of the output profile obtained using the SFAW strategy is conclusively better than any other MIMO anti-windup approach.

We also present the $l_2$ norm of the optimal steady state output profile obtained minimizing (3) subject to the input constraint $u(t) \in U$. This ideal QP solution provides a lower bound for all steady state solutions and can be viewed as a benchmark for the performance. In both cases the SFAW strategy result is very close to the ideal QP solution.

6 Conclusions

In this contribution we analyse some of the properties of a recently proposed anti-windup strategy for high dimensional cross-directional control systems. It has been shown that the described SFAW strategy clearly outperforms two other commonly used MIMO anti-windup approaches, namely input clipping and input scaling. In terms of steady state performance we show that the adaptive algorithm ultimately chooses, for a given disturbance, the spatial directions with feasible control and discards the directions associated to singular values modeled with the wrong sign.
References


