On the Number of Points on Shells for Shifted $Z^{4n}$ Lattices

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Abstract

In this correspondence, we prove a conjecture on the number of points on shells for the shifted $Z^4$ and the shifted $Z^8$ lattices. We also find an expression for the number of points on shells for any shifted $Z^{4n}$ lattice.

List of index terms: integer lattice, theta series, multidimensional signal sets

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1 Introduction

In [1, p. 107], Ruiz notes that not all shells of points on $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ have points on them. However, he conjectures that for the shifted $\mathbb{Z}^4$ and $\mathbb{Z}^8$ lattices, all shells have points on them. In this correspondence we prove, using arguments from number theory, that this conjecture is true. We also extend it to any shifted $\mathbb{Z}^{4n}$ lattice. This result has applications in the design of multidimensional signal sets.

II Shells on the Shifted $\mathbb{Z}^n$ Lattice

In 2 dimensions, the squared radius of a shell of points is given by

$$r_k = 2(k - 1) + 0.5, \quad k = 1, 2, \ldots$$

Thus the first shell has a squared radius of 0.5, the second shell has a squared radius of 2.5 and so on. Not all shells have points on them. For example, the sixth shell with squared radius 10.5 contains no points. In other words there are no pairs $(x, y)$ of half integers such that $x^2 + y^2 = 10.5$.

In 4 dimensions, the squared radius of a shell of points is given by

$$r_k = 2k - 1, \quad k = 1, 2, \ldots$$  \hspace{1cm} (1)

and in 8 dimensions it is given by

$$r_k = 2k, \quad k = 1, 2, \ldots$$  \hspace{1cm} (2)

Ruiz [1, p. 107] has conjectured that all the shells in 4D and 8D contain points.

III Background Material

To prove the conjecture we need facts and theorems from number theory. We will cite these without proofs.
**Theorem 1** Every positive integer is the sum of four squares.

This is known as Lagrange’s Theorem [2]. Using the theory of theta functions, Jacobi vastly generalized Lagrange’s Theorem and, in particular, he proved the following:

**Theorem 2** The number of representations $r_4(n)$ of a positive integer $n$ as the sum of four squares is

$$r_4(n) = 8 \sum_{d|n \& 4|d} d$$

The summation is over all integers $d$ that divide $n$ but that are not divided by 4. Here representations differing only by sign or order are counted as distinct. In other words, $r_4(n)$ is the number of vectors in the integer lattice $Z^n$ such that the sum of the squares of their coordinates is $n$. A good reference is [3].

Theorem 1 will be helpful in proving that all shells contain points, while Theorem 2 will be used to find the number of points on each 4D shell.

We can obtain the number of vectors of squared norm $m$ on a lattice $\Lambda$ from its theta series which is given by [4, p. 45]

$$\Theta_\Lambda(z) = \sum_{x \in \Lambda} q^{x \cdot x} = \sum_{m=0}^{\infty} N_m q^m$$

where $q = e^{\pi i z}$. For our purpose, we can think of $\Theta_\Lambda$ as a power series in $q$.

The theta series of the integer lattice $Z$ is given by

$$\Theta_Z(z) = \sum_{m=-\infty}^{\infty} q^{m^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots$$

which is the Jacobi theta function $\theta_3(z)$.  

The theta series of the shifted integer lattice $\mathbb{Z} + \frac{1}{2}$ is given by

$$\Theta_{\mathbb{Z} + \frac{1}{2}}(z) = \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2} = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots$$

which is the Jacobi theta function $\theta_2(z)$.

These two Jacobi theta functions are related to a third one, $\theta_4(z)$, through the identity

$$\theta_2(z)^4 = \theta_3(z)^4 - \theta_4(z)^4 \quad (5)$$

where

$$\theta_4(z) = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \cdots \quad (6)$$

Powers of Jacobi theta functions are useful since we can obtain the theta series of higher dimensionality lattices from them. For example,

$$\Theta_{\mathbb{Z}^n}(z) = \Theta_z(z)^n = \theta_3(z)^n. \quad (7)$$

This comes from the definition of the squared norm used in (3).

$$N(x) = x \cdot x = x_1^2 + \cdots + x_n^2 = N(x_1) + \cdots + N(x_n).$$

This also applies to the shifted $\mathbb{Z}^n$ lattice which thus has theta series $\theta_2(z)^n$.

**IV Proof of the Conjecture**

With these facts and theorems we can now prove the conjectures for 4, 8 and 4n dimensions.

**Shifted $\mathbb{Z}^4$ Lattice**
From (4) and (7) we have

\[ \theta_3(z)^n = (1 + 2q + 2q^4 + 2q^9 + \cdots)^4 \]
\[ = \sum_{m=0}^{\infty} a_m q^m. \]

Now remembering that the index \( m \) represents the squared norm of a 4-dimensional vector taken from an integer lattice, we have by Theorem 1 that \( a_m > 0, \forall m \). Using (6) we get

\[ \theta_4(z)^n = (1 - 2q + 2q^4 - 2q^9 + \cdots)^4 \]
\[ = \sum_{m=0}^{\infty} b_m q^m \]

where \( b_m = a_m \) for \( m \) even and \( b_m = -a_m \) for \( m \) odd.

By (5) we can obtain the theta series of the shifted \( Z^4 \) lattice from \( \theta_3(z)^4 \) and \( \theta_4(z)^4 \).

\[ \theta_2(z)^4 = \theta_3(z)^4 - \theta_4(z)^4 \]
\[ = \sum_{m=0}^{\infty} a_m q^m - \sum_{m=0}^{\infty} b_m q^m \]
\[ = \sum_{m=0}^{\infty} c_m q^m \]

where \( c_m = 0 \) for \( m \) even and \( c_m = 2a_m \) for \( m \) odd. Since \( m \) represents the squared norm of a 4-dimensional vector we can see that the squared radius of a shell is \( 2k - 1 \), \( k = 1, 2, \ldots \), in agreement with (1) and, furthermore, all shells have points on them. Thus the conjecture for the shifted \( Z^4 \) lattice is proved.

The number of points on the \( k^{th} \) shell of the shifted \( Z^4 \) lattice are given by the coefficient of \( q^m \), \( m = 2k - 1 \) of the theta series of the lattice. By Theorem 2 and the preceding result we have

\[ c_m = 2a_m \]
\[ = 16 \sum_{d|m \text{ and } 4|d} d. \]  \tag{8}

**Shifted \( Z^8 \) Lattice**

The theta series of the shifted \( Z^8 \) lattice is obtained by squaring the theta series of the shifted \( Z^4 \) lattice, i.e.

\[
\theta_2(z)^8 = (\theta_2(z)^4)^2 = \left( \sum_{m=1, m \text{ odd}}^{\infty} c_m q^m \right)^2
= q^2 \left( \sum_{m=1, m \text{ odd}}^{\infty} c_m q^{m-1} \right)^2
= q^2 \left( \sum_{m=0, m \text{ even}}^{\infty} c_{m+1} q^m \right)^2
= \sum_{m=2, m \text{ even}}^{\infty} d_m q^m.
\]

Note that \( d_m > 0, \forall m \geq 2 \) and \( m \) even. Squaring a series is equivalent to multiplying the series by itself. If we multiply two power series containing only even powers we obtain a series with only even powers. Furthermore, since this series contains all even powers (including 0) and all its coefficients are positive, the squared series will also contain all even powers.

The squared radius of the \( k^{th} \) shell is given by \( 2k, k = 1, 2, \ldots \), in agreement with (2) and all shells contain points. Thus the conjecture is proved for 8 dimensions.

Even though there are theorems such as Theorem 2 for \( n \) other than 4 [4, p. 108], we cannot find an expression for the number of points on a shell of the shifted \( Z^8 \) lattice because there is no identity such as (5) for \( n = 8 \). However, it is still relatively easy to compute the theta series of this lattice by squaring the theta series of the shifted \( Z^4 \) lattice. Table I gives the first 10 terms of the theta series of the shifted \( Z^4 \) and \( Z^8 \) lattices, or equivalently the squared radii for the first 10 shells. The theta series of the shifted \( Z^4 \) lattice was computed using (8).

The reader can compare this table to the tables in [1, pp. 100–103] obtained via a computer search and verify that they are equivalent.
Shifted $Z^{4n}$ Lattice

By a development similar to the above we can find that

$$\theta_2(z)^{4n} = (\theta_2(z)^4)^n$$

$$= \sum_{m=n, m \text{ even}}^{\infty} e_m q^m.$$ 

Again we have $e_m > 0$, $\forall m$ even after an initial value which, in this case, is $n$. Note that the case of $n = 0$ is the $Z + \frac{1}{2}$ lattice which, trivially, has two points on all its shells. Thus the conjecture is true for all values of $n$.

V Summary

We have proved a conjecture regarding the number of points on a shell for the shifted $Z^4$ and $Z^8$ lattices. Furthermore, the results were extended to any shifted $Z^{4n}$ lattice ($n = 0, 1, \ldots$). These results provide an easy way to compute the number of points on shells for the type of lattices used in the design of multidimensional signal sets [5] or in vector coding [6].

References


Table I

Theta series of the shifted $\mathbb{Z}^4$ and $\mathbb{Z}^8$ lattices

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Table caption

Table I: Theta series of the shifted $Z^3$ and $Z^8$ lattices