Fourier Transform deconvolution of noisy signals and partial Savitzky-Golay filtering in the transformed side

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Abstract

The use of Fourier transform for deconvolution of noisy signals and the problem of the division in the transformed side are briefly discussed. Comparison between many deconvolutions of noisy signals with and without partial Savitzky-Golay filtering of the transformed functions is presented. Simulations with analytically known functions show that a proper deconvolution method can produce correct result, at least for moderately noisy functions.

Introduction

The response $D(t)$ of a system, having the characteristic function $I(t)$ (impulse response function), when driven by an input function $E(t)$ is given by the convolution of $I(t)$ by $E(t)$, defined as follows:

$$D(t) = \int_0^t I(\tau)E(t-\tau)d\tau \qquad (1)$$
$$= \int_0^t I(t-\tau)E(\tau)d\tau$$
$$= I(t) \cdot E(t) = E(t) \cdot I(t)$$

Obtaining $D(t)$ from $I(t)$ and $E(t)$ is straightforward. If the only available signals or functions are $I(t)$ and $D(t)$, extracting $E(t)$ from them entails the so-called deconvolution process. Direct extraction of $E(t)$ from the integrals in Eq. 1 is not so easy. Fortunately, once the functions are Fourier transformed, the operation becomes simple, Eq. 1 can be written as:

$$F\{D(t)\} = F\{I(t) \cdot E(t)\} \qquad (2)$$
$$= F\{I(t)\}F\{E(t)\}$$

Hence, the deconvolution can be performed as follows:

$$E(t) = F^{-1}\left\{ \frac{F\{D(t)\}}{F\{I(t)\}} \right\} \qquad (3)$$

At first glance, using Eq. 3 for deconvolution seems straightforward and easy. Nevertheless, since Eq. 3 involves a division, major difficulties appear when $D(t)$ and $I(t)$ are digitized and noisy signals. If, particularly the denominator function, departs from the correct analytical value, particularly for its values near to zero, the deconvoluted signal will be greatly distorted. These departures may have two sources: the inaccuracies inherent to the discrete Fourier transform (DFT), and noise. As discussed elsewhere [2], we have developped a very accurate method to approach almost as much as one wants the exact analytical Fourier transform of a function from its digitized version. So the problem of inaccuracies due to DFT may be considered as cleared. The second difficulty is due to the noise.

Deconvolution of noisy signals

A division of a function, noisy or not, by another noisy function gives a noisy result. When the denominator function gets close to zero, there is almost an “explosion” of noise in the result of the division. Hence, it is often said, and sometimes written [3], that the division in the transformed side generates noise. One should prefer the more correct formulation of [1]: “... the presence of this noise may give rise to a mathematically exact result that has no physical significance.” Fourier transform deconvolution is then often considered as impraticable. Nevertheless, it remains attractive because of its straightforwardness. Let us look, through few examples, at the performances of the Fourier transform for deconvolution and, at an attempt to solve, at least partially, that noise problem.

Defining the characteristic function[1] of a system as,

$$I(t) = \begin{cases} e^{at}[b_1\cos(2\pi f_1 t) + b_2\sin(2\pi f_2 t)] \\ \qquad \text{for } t \in [0, T] \\ 0 \quad \text{elsewhere} \end{cases} \qquad (4)$$

and the input function as,

$$E(t) = \begin{cases} cte^{dt} & \text{for } t \in [0, T] \\ 0 & \text{elsewhere} \end{cases} \qquad (5)$$

one can compute the convolution of Eq. 4 and Eq. 5 to obtain the somewhat elaborated expression for $D(t)$ expressed in Eq. 6.

1. We use one dimension causal functions and the convolution integral between the limits 0 and t. The ideas discussed here remain valid for n-dimension non-causal functions between $-\infty$ to ∞.

$$
\begin{aligned}
&\omega_1 = 2\pi f_1 \quad , \quad \omega_2 = 2\pi f_2 \quad , \quad \tau = a - d \\
&A_1 = \tan\left(\frac{1}{2}\omega_1 t\right) \quad , \quad A_2 = \tan\left(\frac{1}{2}\omega_2 t\right) \\
&B_1 = \omega_1^{\,2} + \tau^2 \quad , \quad C_1 = \omega_1^{\,2} - \tau^2 \\
&B_2 = \omega_2^{\,2} + \tau^2 \quad , \quad C_2 = \omega_2^{\,2} - \tau^2 \\
&D_1(t) = \frac{(C_1 A_1^{\,2} + 4\omega_1 \tau A_1 - C_1) b_1 c e^{at}}{(1 + A_1^{\,2}) B_1^{\,2}} + \frac{(C_1 - \tau B_1 t) b_1 c e^{dt}}{B_1^{\,2}} \\
&D_2(t) = \frac{2(\tau\omega_2 A_2^{\,2} - C_2 A_2 - \tau\omega_2) b_2 c e^{at}}{(1 + A_2^{\,2}) B_2^{\,2}} + \frac{\omega_2 (B_2 t + 2\tau) b_2 c e^{dt}}{B_2^{\,2}} \\
&D(t) = D_1(t) + D_2(t)
\end{aligned}
\tag{6}
$$

In the present example, the following numerical values for parameters will be used: $a = \ln(10^{-3})$, $b_1 = 1$, $b_2 = 0.25$, $f_1 = 100$, $f_2 = 2000$, $c = 2x10^5$, $d = 2a$ and $T = 1$.

Now, let us use 25 functions $I(t)$ and 25 functions $D(t)$ as previously defined by Eq. 4 and Eq. respectively but, this time, with Gaussian noise added to each of them. The standard deviation (σ) of the noise is set to 0.0033 so that the peak amplitude (3σ) of the noise is approximately 1% of the maximum value of the functions. On the average, on all the interval $t \quad [0 , T]$ it gives a noise of 2.8% on $I(t)$ and 4.7% on $D(t)$. The functions $I(t)$ and $D(t)$ are equidistantly digitized for a number of points $N = 16384$. We want to recover the function $E(t)$ from numerical deconvolution of the noisy functions $D(t)$ and $I(t)$.

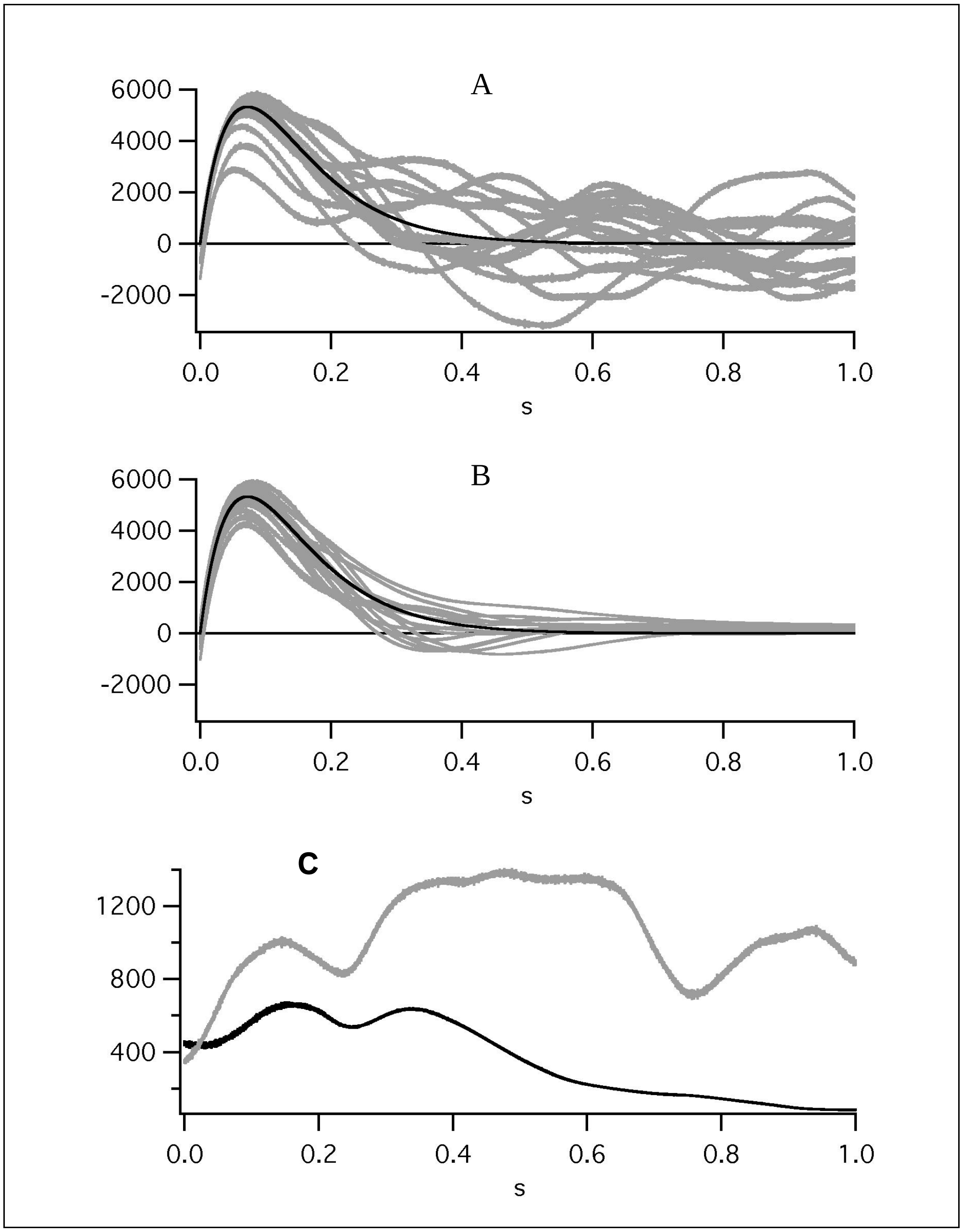
First, the deconvolutions are numerically performed according to Eq. 3. Fig. 1-A shows the results. The black curve is the exact resul given by Eq. 5. The grey background is the stacking of the 25 results obtained by the deconvolution of the 25 pairs of the noisy functions $I(t)$ and $D(t)$. Second, the 25 deconvolution are again numerically performed according to Eq. 3, but this time a Savitzky-Golay filter of order 2 with a width of 25 points has been partially applied on the transformed functions. (The author has developped a Maple[2] program that gives the Savitzky-Golay coefficients for any order and any number of points, symmetrically or asymmetrically. One can find theory, examples and further references about Savitzky-Golay filters in [4]. In the present paper we have used the Savitzky-Golay filter provided by the software Igor from WaveMetrics)

The transformed functions exhibits narrow peaks. Applying the filter everywhere would have clipped these peaks and then distorted the results. Since the values of the transformed functions in the peaks are not close to zero, the result of the division is not very sensitive to noise, so there is no needs to filter in these regions. We have merely applied the filter when the modulus of the

2. Maple is a powerfull symbolic and numerical math. software, by Waterloo Maple Software.

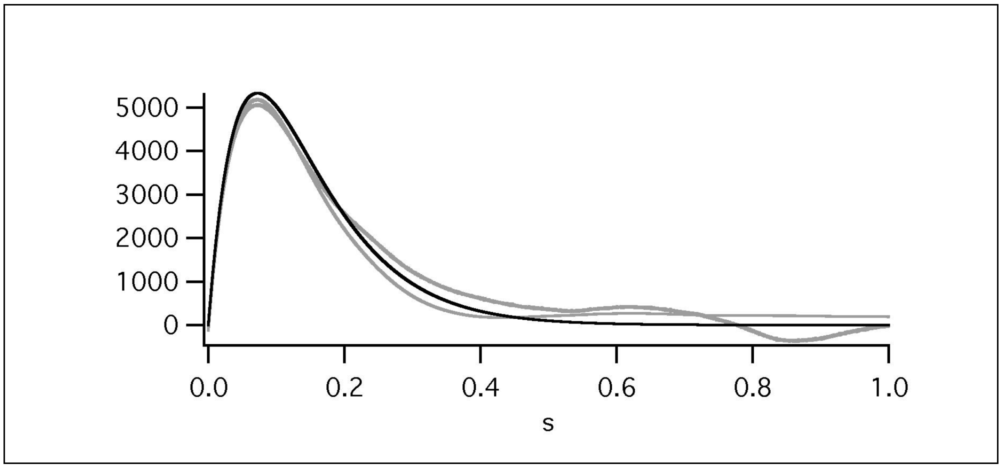
denominator function ($FI(t)$) was one tenth of the modulus of its characteristic peaks. The results are shown in Fig. 1-B. Grey curve of Fig. 1-C shows the standard deviation function of the 25 deconvolutions without filtering. Black curve of Fig. 1-C shows the standard deviation function of the 25 deconvolutions with filtering. On the average, the standard deviation function with filtering is about 3 times smaller than the one without filtering. It is not a dramatic but a significant improvement.

Fig. 1. Deconvolutions of noisy functions.



The black curve of Fig. 2 is the exact result $E(t)$. The slightly oscillating grey curve of Fig. 2 is the average of the 25 deconvolutions without filtering. The other grey curve of Fig. 2 is the average of the 25 deconvolutions with filtering.

Fig. 2. Average of deconvolutions.



Conclusion

First, as shown by curves in Fig. 1, an adequate limited filtering can significantly reduce the dispersion of results, without, as do inevitably any filter, introducing too much distortion. Second, the results of the average of deconvolutions displayed by Fig. 2 show that Fourier transform deconvolution of noisy signals is not impraticable. Increasing the number of deconvoluted pairs of signals or increasing the number of system's characteristic signals $I(t)$ to be deconvoluted with a given numbers (possibly only one) response signals $D(t)$ would indeed improve the averaged result. We can even conjecture that paradoxically, designing a proper filter in the transformed domain could be more efficient in two or more dimensions than in one because each filtered point is surrounded by much more than two immediate neighbours.

Bibliography

[1] J. C. Andre, L. M. Vincent, D. O'Connor and W. R. Ware. Applications of Fast Fourier transform to Deconvolution in Single Photon Counting. J. of Phys. Chem. Vol. 83, No. 17, pp.2285-2294, 1979

[2] N. Beaudoin. A high accuracy mathematical and numerical method for Fourier transform, integral, derivatives and polynomial splines of any order. Canadian Journal of Physics. Vol. 76, No. 9, pp.659-677, September 1998.

[3] J. N. Demas. Excited States Lifetime Measurements. Academic Press, New York, 1983.

[4] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery. Numerical Recipes in Fortran. Cambridge, 1992.